Symplectic geometric construction of cluster variables in braid varieties (joint work with Roger Casals)

BIRS, Representation theory, symplectic geometry and cluster algebras

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Outline

- Given a manifold M whose first Betti number is b, the set of complex rank 1 local systems on M forms the space $(\mathbb{C}^\times)^b$. Considering iterations of the monodromy along a 1-cycle $L \mapsto m_\gamma(L)$ gives regular functions (holomorphic functions) on the space.
- Given a symplectic manifold M, we can study the space of complex rank 1 local systems on Lagrangian submanifolds $L \subset M$. This is a generalization of the previous situation (where we can consider $M = T^*X$ and $L = X$). The goal of the talk is to explain how to construct regular functions on such moduli spaces.

Legendrians and Lagrangian fillings

Definition

A contact manifold (Y, ξ) is a $(2n+1)$-dimensional manifold with a maximally non-integrable hyperplane distribution ξ. A Legendrian submanifold $\Lambda \subset (Y, \xi)$ is an n-dimensional submanifold such that $T\Lambda \subset \xi$.

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Definition

Let $\Lambda \subset (Y, \ker \alpha)$ be Legendrian submanifolds. An exact Lagrangian filling L of Λ is a properly embedded exact Lagrangian submanifold $L \subset (\mathbb{R}_+ \times Y, d(s\alpha))$, such that

- when s is large, $L \cap (s, +\infty) \times Y = (s, +\infty) \times \Lambda$;
- the primitive f_L such that $df_L = s\alpha|_L$ is a constant on $(s, +\infty) \times \Lambda$.

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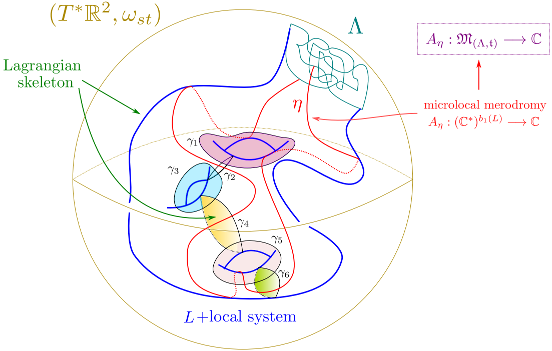
Theorem (Ekholm–Honda–Kálmán, Pan, Shende–Treumann–Williams–Zaslow, Casals–Gao, Gao–Shen–Weng, Casals–Ng, etc.)

For a large class of Legendrian (-1)-braid closures $\Lambda(\beta\Delta)$ of the braids $\beta\Delta$, there exist (infinitely) many exact Lagrangian fillings up to Hamiltonian isotopies relative to the boundary.

Lagrangian fillings and disk surgeries

Theorem (Shende–Treumann–Williams–Zaslow, Casals–Zaslow, Casals–Weng, Casals–Gao, etc.)

Lagrangian fillings of these Legendrian links (in some cases) can be constructed by Lagrangian surgeries of a given filling L along an embedded Lagrangian disk D which intersects with the filling L along ∂D.



Moduli space of Lagrangian fillings

- Given a Legendrian knot/link Λ, one can construct the augmentation variety (Ng) of the Legendrian contact homology (Chekanov, Eliashberg), or the derived moduli stack (Toën–Vaquié) of microlocal rank 1 sheaves with singular support on the Legendrian $\mathcal{M}_{fr}(\Lambda)$.

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- For braid positive Legendrian knots $\Lambda = \Lambda(\beta)$, the derived moduli of microlocal rank 1 sheaves is a global quotient $[M_{fr}(\Lambda)/\mathbb{C}^\times]$, where $\mathbb{C}^\times$ acts trivially and $M_{fr}(\Lambda)$ is isomorphic to the augmentation variety (Kálmán, Casals–Gorsky–Gorsky–Simental, Casals–Ng).

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- An exact Lagrangian filling L with $b_1(L) = b$ defines an embedding $[(\mathbb{C}^\times)^b/\mathbb{C}^\times] \subset \mathcal{M}_{fr}(\Lambda)$ or $(\mathbb{C}^\times)^b \subset M_{fr}(\Lambda)$ (Jin–Treumann). We thus view it as the moduli of Lagrangian fillings with local systems.

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- Complete flags in $\mathbb{C}^n$ are parametrized by G/B for $G = GL(n, \mathbb{C})$, and $M_{fr}(\Lambda) = \{(F_0, \ldots, F_r) \in (G/B)^{r+1} \mid F_0 \xrightarrow{s_{i_1}} \ldots \xrightarrow{s_{i_r}} F_r, F_0 = B, F_r = w_0 B\}$. Namely, F_i and F_{i+1} are in relative position $s_i \in W = S_n$ and $w_0 \in W = S_n$ is the longest word.

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Theorem (Casals–Gorsky–Gorsky–Le–Shen–Simental, Galashin–Lam–Sherman-Bennet–Speyer)

The moduli space $M_{fr}(\Lambda) = X(\beta)$ is a cluster A-variety.

Moduli space of Lagrangian fillings

- The cluster variables are simply the standard coordinates on the torus charts $(\mathbb{C}^\times)^b$ (Shende–Treumann–Williams–Zaslow, Casals–Weng). In other words, when restricted to an torus chart of a Lagrangian filling, they are just holonomies along 1-cycles.

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Question

Given the moduli space of Lagrangian fillings $M_{fr}(\Lambda)$ of a Legendrian link Λ and a Lagrangian filling L which defines an open subset $(\mathbb{C}^\times)^b \subset M_{fr}(\Lambda)$, when does a holonomy function extend globally to $M_{fr}(\Lambda)$?

Our main theorem

Theorem (Casals–L.)

Let $\Lambda \subset S^3_{st}$ be a (pointed) Legendrian link and $L \subset \mathbb{R}^4_{st}$ an exact Lagrangian filling with embedded Lagrangian disks $\mathscr{D}$ intersecting with L along their boundary (which span the 1st homology of L).
Then, any (relative) 1-cycle η in L that intersects **positively** *with the boundary of Lagrangian disks $\mathscr{D}$ defines a regular function on the derived moduli stack $\mathcal{M}_{fr}(\Lambda)$ that coincides with the microlocal holonomy along η when restricted to the algebraic torus defined by L.*

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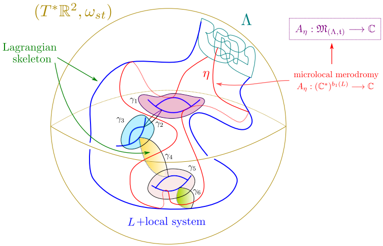

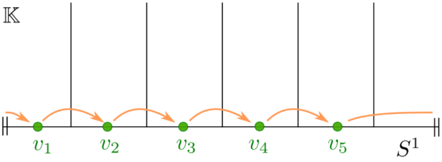

- We view objects in the moduli space as microlocal sheaves on the Lagrangian skeleton (Nadler, Shende, etc.). When restricting along the 1-cycle, we can see that the Lagrangian disk becomes vertical lines and they obstruct the parallel transport maps.

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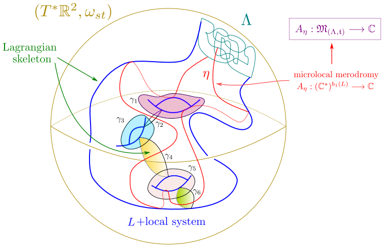

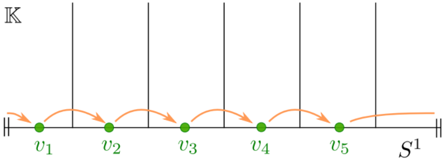

- We view objects in the moduli space as microlocal sheaves on the Lagrangian skeleton (Nadler, Shende, etc.). When restricting along the 1-cycle, we can see that the Lagrangian disk becomes vertical lines and they obstruct the parallel transport maps.
- Our proof constructs a Hochschild homology class (which should be a version of symplectic cohomology) corresponding to the closed positive loop and shows that it induces the holonomy along the cycle.

Intuition from mirror symmetry

- Our result does not require the Lagrangian skeleton to come from a braid-positive Legendrian, does not require the skeleton to be 2-dimensional, and does not require the moduli of sheaves to have (microlocal) rank 1.

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- Our result does not require the Lagrangian skeleton to come from a braid-positive Legendrian, does not require the skeleton to be 2-dimensional, and does not require the moduli of sheaves to have (microlocal) rank 1.
- From a mirror symmetry perspective, heuristically, one should interpret a regular function $M(\Lambda) \to \mathbb{C}$ on the B-side through inclusions of the Liouville sector $\mathbb{C}^\times_{Re\leq 1} \times \mathbb{C}^{n-1}_{-1\leq Re\leq 1} \hookrightarrow (X, \Lambda)$ on the A-side $\mathbb{C}^\times_{Re\leq 1}$ (equivalently $\mathbb{C}^\times, W = z$) is exactly the mirror of $\mathbb{C}$.

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- Let $\Lambda \subset S^3_{st}$ be a (pointed) Legendrian knot. Then, there exists a holomorphic symplectic structure on $M_{fr}(\Lambda)/(\mathbb{C}^\times)^r$.
- When Λ admits an oriented graded exact Lagrangian filling L with embedded Lagrangian disks $\mathscr{D}$ intersecting with L along their boundary (which span the 1st homology of L), then the symplectic form restricts to the intersection form on L.

Thank you!